Continuous variations of the waterbomb base tessellation

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Abstract

Ten families, of dimension up to five, of flat foldable continuous variations on the waterbomb base tessellation are given. A relationship between certain flat tessellations and corrugations, and the two colouring of the crease pattern is explained.

1 Introduction

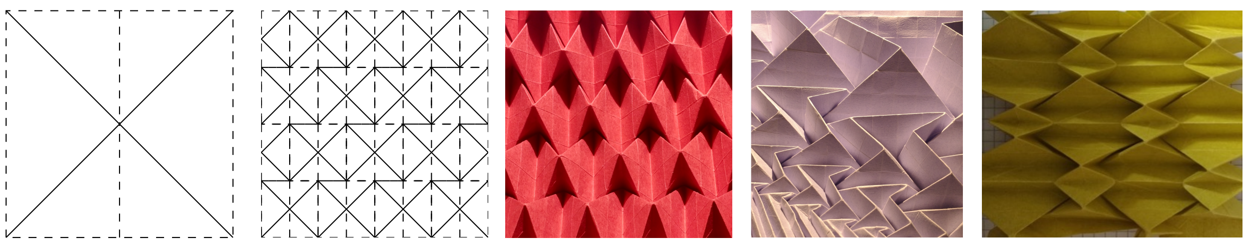

Figure 1: *The waterbomb crease pattern (1); tessellated by Resch (2), folded (3). Variations (4), (5).*

An *origami tessellation* is invariant under translations in two directions. Described by Momotani [12] and Fujimoto[8] in the 1980's, similar fabric designs go back to the 1800's [14], [17]. For theory and examples see e.g., Lang [6] and e.g., [10],[11], [13]. My motivating question is the relationships between origami tessellations. Given a flat folding origami tessellation, how can it be continuously varied, while remaining a flat foldable tessellation? Given two continuous variations, can they be combined to form a larger family of tessellations? When is it possible to continuously vary one tessellation to another? This paper addresses variations on the waterbomb tessellation, extending Lamiori tessellations [5]. The waterbomb tessellation, Figure 1, conceived by Resch has been utilised art, design, architecture, engineering [3], [2], [7].

2 Seven families of waterbomb base origami tessellations

Parameterised origami tessellations, particularly those based on Archimedian tilings, have been given by [1], [4], [6] and others. Lamio [5] describes variations on the waterbomb tessellation. Further varying structure produces new origami tessellations. Three of these families are given as interactive JavaScript programs at [15]. The crystallographic flat origamis and the waterbomb tessellations are both flat, but the later belong to the category "origami corrugations". In this paper, we pass continuously from flat tessellation to corrugation type, described in more detail in § 5. Criteria and definitions for flat foldability are described in [9]. The most easily checked criterion, that the sum of alternate angles at any vertex is 180°, can be verified by inspection of the crease patterns given in this paper. Foldability is shown by folding, illustrated by photographs.

In Figure 2, the x-axis corresponds to the Lamiori variations [5]. The y-axis corresponds to variations of the "dragon scales" tessellation. Can these can be combined to form a 2-dimensional family of variations? A solution is shown as Family B in Figure 3, also showing two other families. The units in the top row of Figure 3 tessellate as shown in the lower row. Further families of variations are shown in Figures 4 and 5.

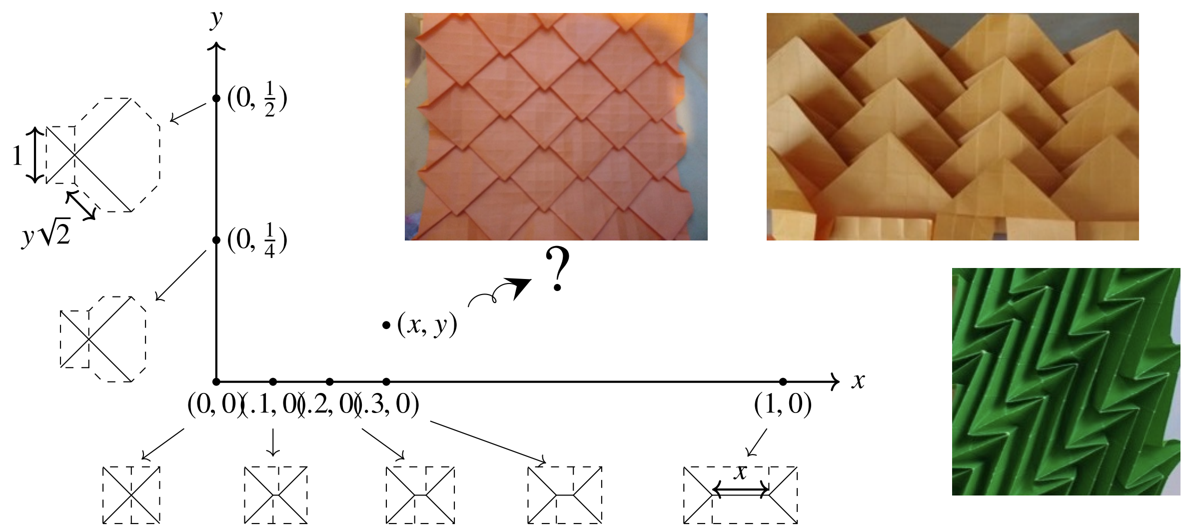

Figure 2: *Points $(x, 0)$ and $(0, y)$ along the axes correspond to known variations of the waterbomb tessellation. What tessellation might a point (x, y) correspond to? (See Figure 3 B.)*

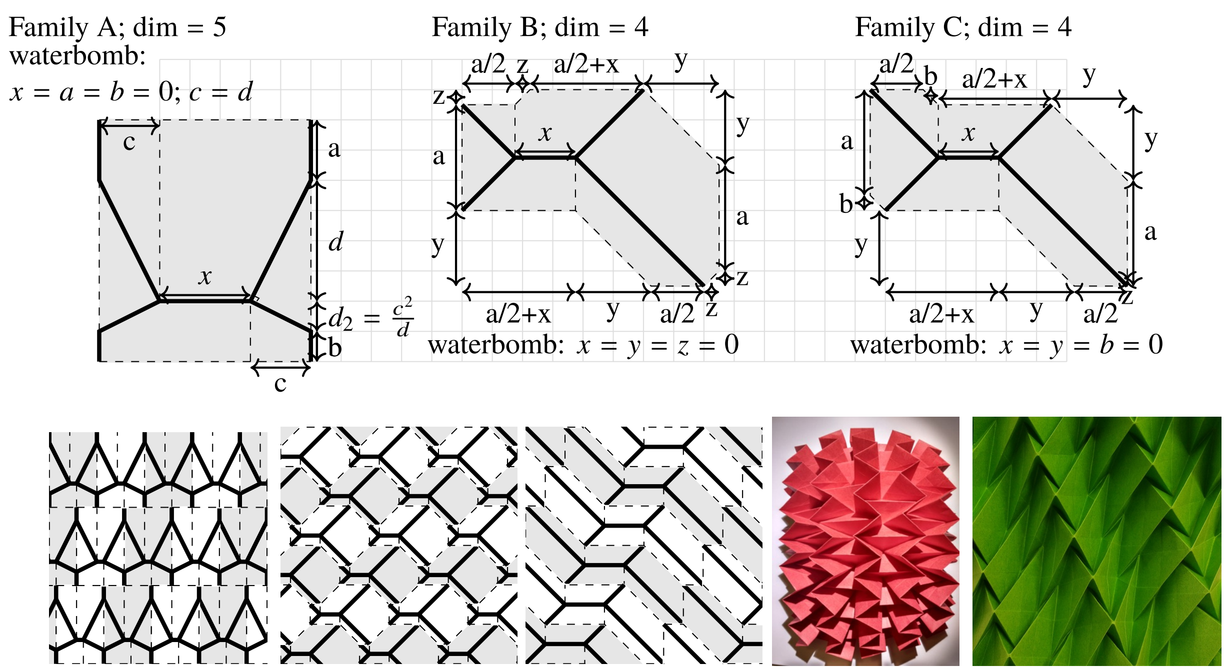

Figure 3: *units for tessellations based on variation of the waterbomb tessellation. Families B and C coincide when $b = z = 0$. Parameter values for the original waterbomb tessellation are shown. Tessellations of these units are shown in the lower row, with folded examples of them.*

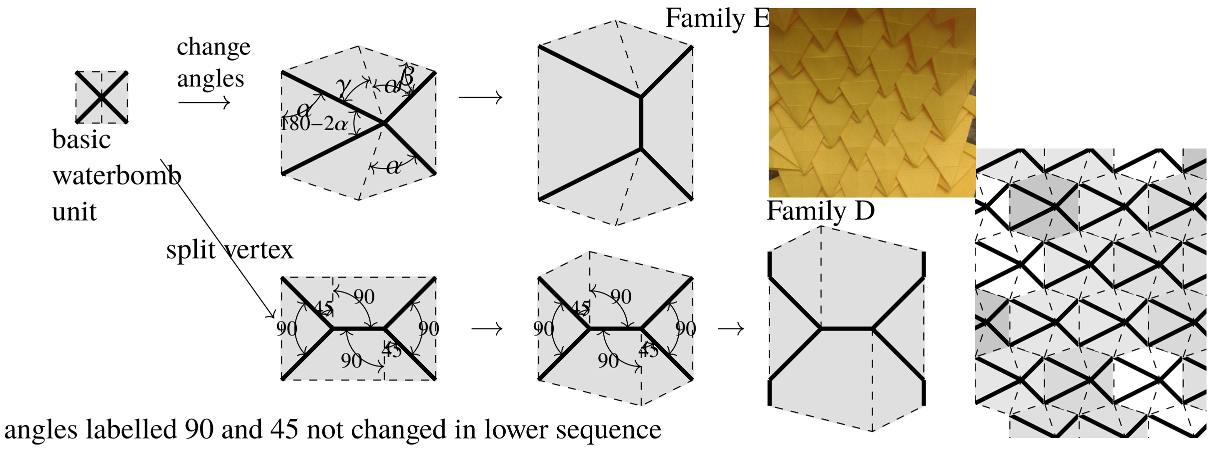

Figure 4: *Families with varying angles. Tessellation and folded origami of a member of family E shown.*

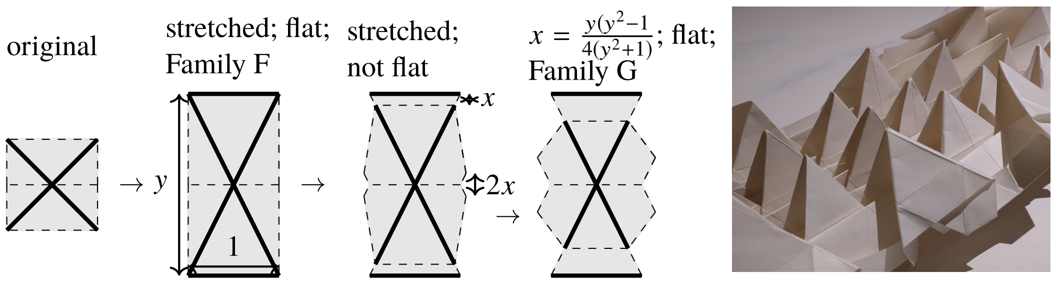

Figure 5: *Stretching unit, and splitting a crease. An extreme case, with angles* 60° *is shown folded.*

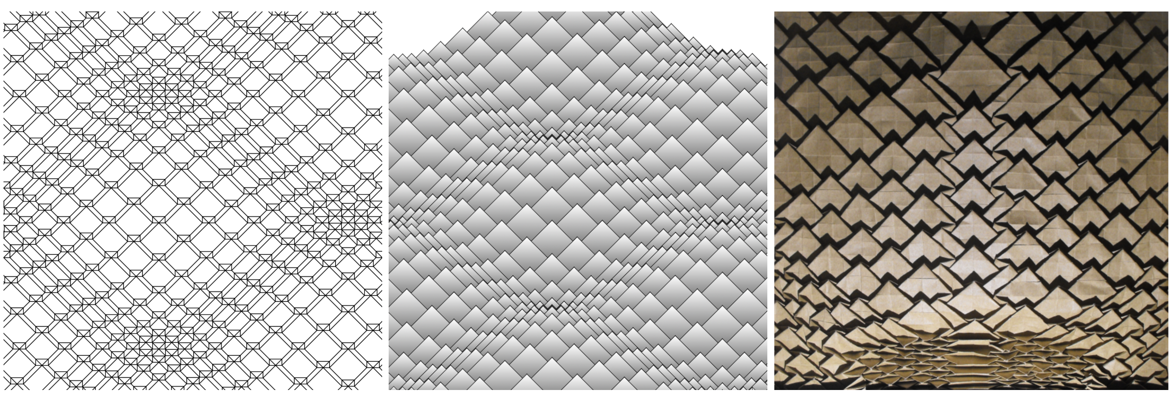

Figure 6: *Tessellations obtained from units as in Figure 3 via process in Figure 7. Crease diagram on left results in middle folded origami; another similar construction results in the figure on the right.*

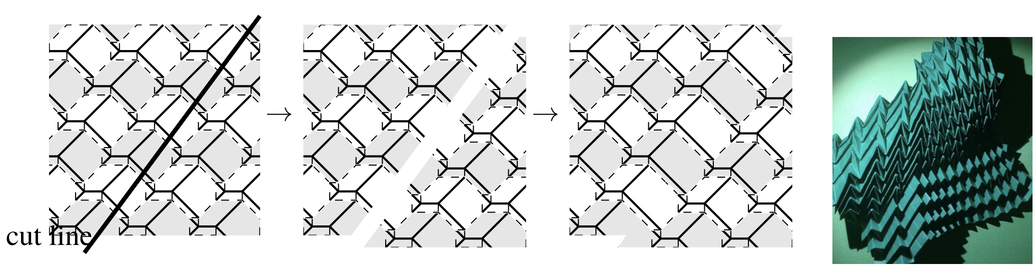

Figure 7: *Modification of a tessellation from Figure 3, by cutting through parallel lines.*

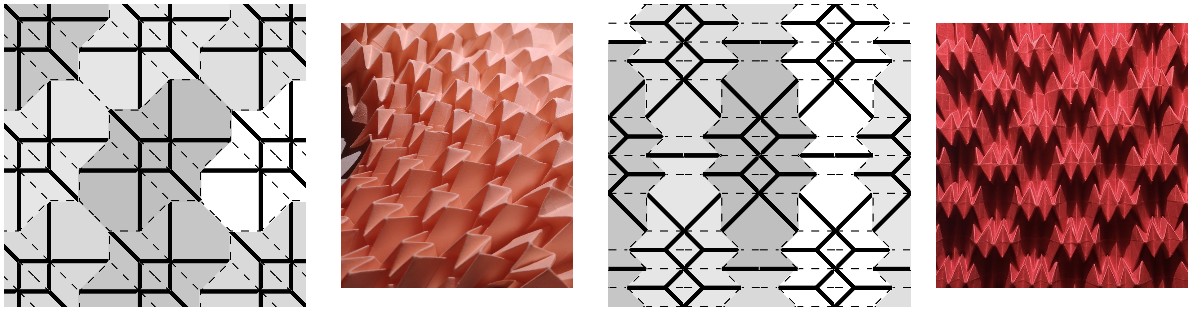

Figure 8: *Families H and I: The single peak of the waterbomb base can be continuously replaced by a double or triple peak, and so on. These are examples of the family shown in Figure 9*

3 Methods of constructing families

Several techniques are used to obtain these families: split vertex; stretching; cut and glue; change angles; splitting an edge. Most of the examples in Figure 3 only split the peak vertex and change lengths, not angles. Angles are changed in Families in Figures 4 and 5. In Figure 5, the origami is first stretched. Generally stretching will not preserve flatness, but this case works, and the flat folded stretched version will have a curved profile. The modification to Family G results in a straight profile. The parameter in Figure 5 is limited to a small enough range to ensure the overlapping condition of flat foldability is satisfied. Family G is obtained from Family F by splitting a valley fold into a union of valley/mountain/valley folds. in the limit as the parameter x tends to 0, the union of the three folds becomes a single valley fold. In replacing the valley fold by the three parallel creases, width x apart, the resulting create pattern is not flat foldable. However, there is only one vertex for which the sum of alternate angles is not in general 180°. A calculation shows that as x is varied, the angle condition will be satisfied when $x = \frac{y(y^2-1}{4(y^2+1)}$. The crease patterns in Figure 3, bottom row, contains infinite sets of parallel line segments. If we cut along a line only intersecting these line segments, and move the two halves of the paper away from each other in the direction of these parallel lines, then reconnect the parallel lines, we obtain a tessellation consisting of units of the same family, but with different parameters. This process is shown in Figure 7. The process can be repeated. Some resulting crease patterns and folded pieces are shown in Figure 6. Some areas of this pattern are the regular waterbomb tessellation; some are the dragon scales tessellation. This shows how to incorporate both in the same pattern. Also it shows how some parts of the same pattern are more corrugation like, and some are more flat tessellation like. Different parameters of the unit produce a different density of the folding, as described in § 5, which results in a three dimensional effect in the folded piece, as in Figure 7. The families are related; e.g., Family D in Figure 4 extends Family A of Figure 2. The different splittings of a vertex, in perpendicular directions in Figure 5, families D and E, could be combined to form a non-flat folding variant of the waterbomb tessellation.

4 multiple peak units

So far, each unit has a single peak or ridge. In Figure 3, the peak when $x = 0$ becomes a mountain ridge when $x > 0$, giving the Lamiori patterns [5]. It is also possible to construct units with several peaks. Figure 8 shows units with 2 and 3 peaks. This can be extended to longer strings of different sized peaks as in Figure 9. In Figure 10 units of two different sizes are combined in one tessellation. The deformation of family B, Figure 2 can be applied to each unit, resulting in the pattern on the right of Figure 10. This process has transformed a corrugation into a spread out flat tessellation. This is discussed more in § 5. Similar constructions give fractal families of waterbomb units, one of which is shown in Figure 1.

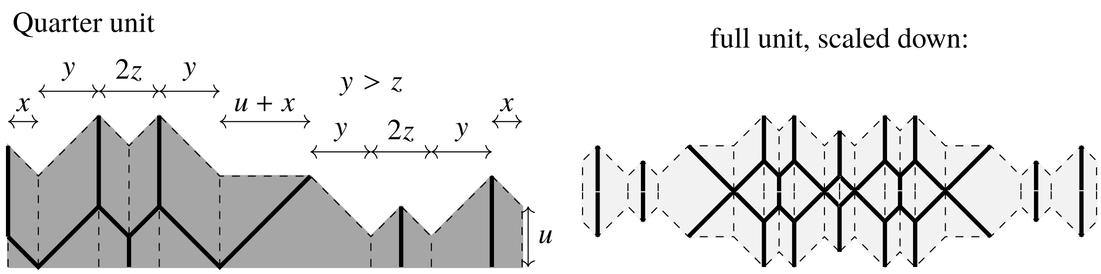

Figure 9: *Waterbomb variation units with multiple peaks. Taking $x = y = z = 0$ gives the usual waterbomb tessellation. Families H and I of Figure 8 are obtained with $y = z = 0$ and $x = z = 0$ respectively. This four peak unit can be extended to have arbitrarily many peaks.*

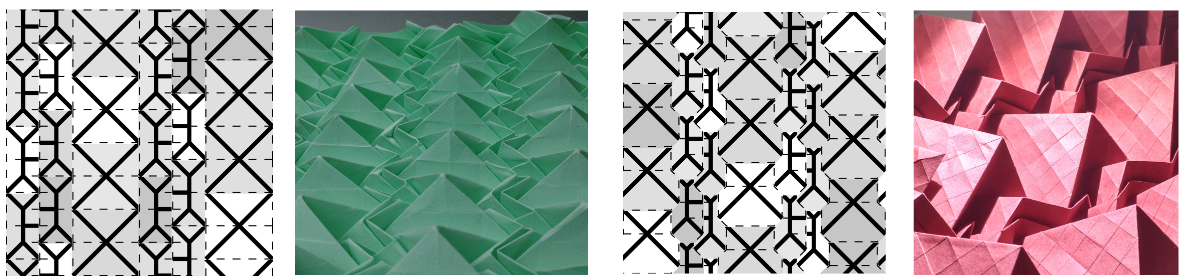

Figure 10: *Family J: Variation of Family B, Figure 3 applied to a tessellation of different sized waterbomb units.*

5 Corrugation and tessellation

In this paper, all the tessellations are flat foldable. There are two distinct types. The resulting flat folded origami is either a finite to one, or an infinite to one projection to the plane in which the resulting flat folded origami lies. In the case of a finite to one projection, we generally fold the paper completely flat, and view in the direction of the projection p_2 in Figure 11. These kinds of tessellations are often viewed with light passing through, layers of paper creating a pattern. In the other case, we generally open the origami a little, and view in a perpendicular direction p_1 in Figure 11. For these kinds of tessellations, we often are interested in the effect of the shadows and light bouncing off the paper, rather than looking through the paper. The image in the infinite to one case may be an irregular shape, rather than a strip. Given a flat fold origami tessellation crease pattern, how can we tell which case we are in without folding?

I now define two ratios associated with the pattern. Ratio r_1 is the ratio, in the limit as the paper size tends to infinity, between the folded paper area and the initial paper area. For example, in two simple tessellations in Figure 11 ratios are $0 = 1 : \infty$ and $\frac{1}{3} = 1 : 3$. Note that since the viewable paper can only decrease in size, $0 \leq r_1 < 1$. Every flat foldable origami tessellation crease pattern is two colourable [9]. The ratio r_2 is the ratio between the two areas when the crease pattern is two coloured. These values are $1 : 1$ and $2 : 1$ in the examples in Figure 11. In general they are related as follows:

Theorem.

1. If the crease pattern is for a corrugation, as defined above, then the ratio r_1 is $0 : 1$ and the ratio r_2 is $1 : 1$, and $r_1 = 0 \leftrightarrow r_2 = 1$.
2. The ratios are related by

$$r_2 = \left|\frac{1 + r_1}{1 - r_1}\right|$$

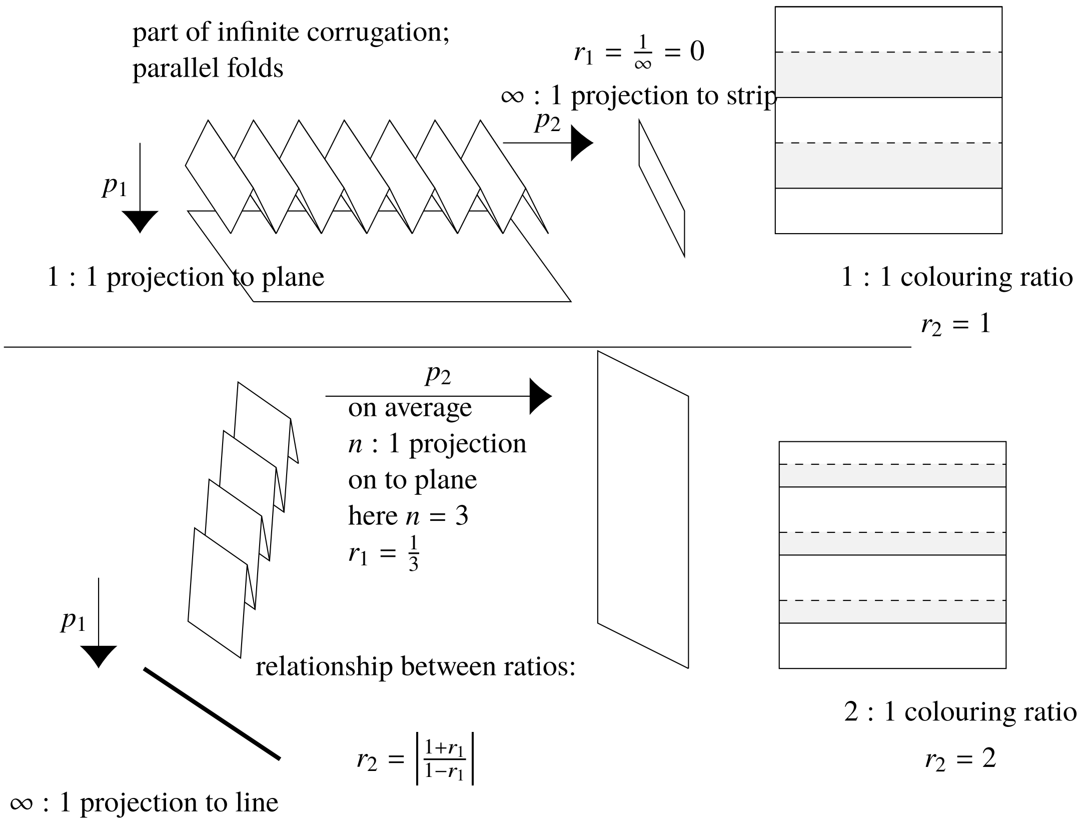

Figure 11: *projections for viewing origami corrugations and tessellations and two coloured crease patterns.*

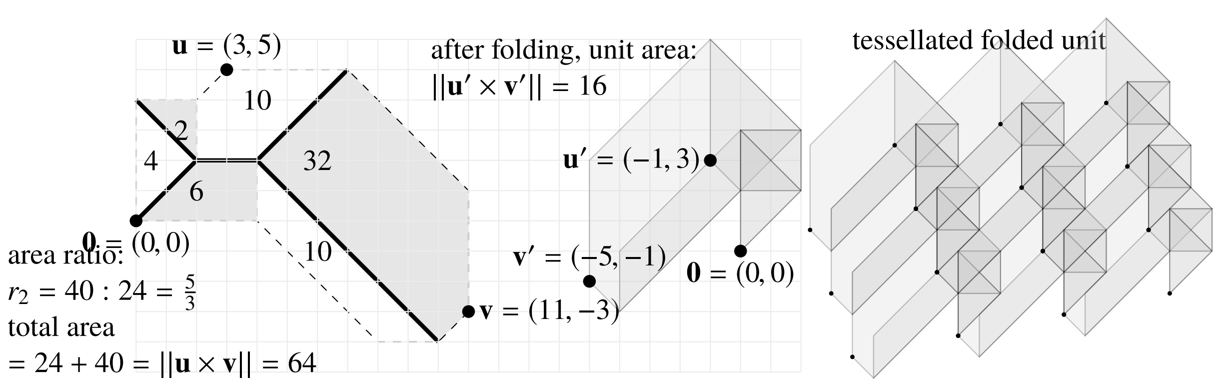

Figure 12: *Example of ratio relationship for waterbomb variation. This is a unit of Family B, Figure 3, with $a, x, y, z = 4, 2, 4, 1$. The ratio of before to after fold area is $r_1 = 16 : 64 = \frac{1}{4}$*

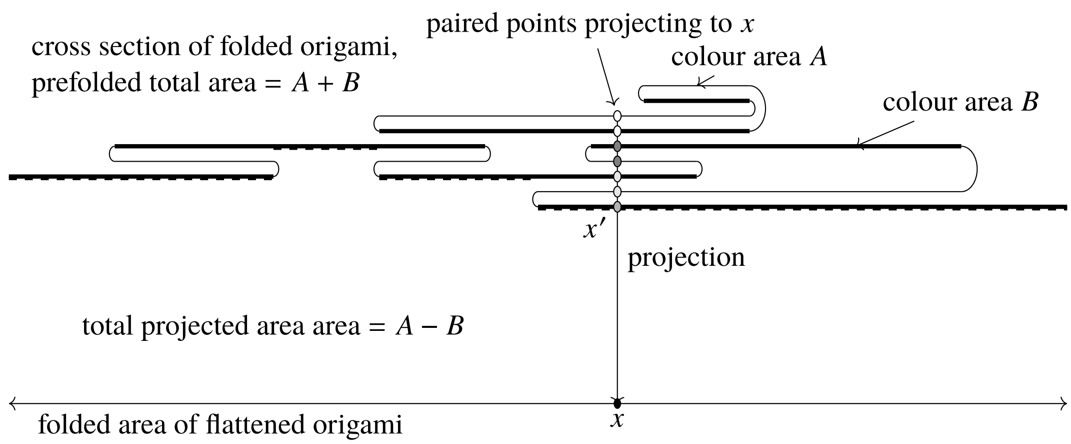

Figure 13: *Cross section of folded tessellation as explanation of relationship between ratios in Figure 11. The non paired parts of the origami are underlined with dashed lines. These are in bijective correspondence with the projected area below.*

This formula was discovered based on studies of variations on the square twist pattern [16]. An example for a waterbomb type unit is give in Figure 12. Note that it is easiest to compute the ratio r_1 by comparing the ratios of a fundamental unit spanned by two vectors generating the lattice under which the tessellation is invariant; these can be seen in the figure.

Proof. Given a crease pattern of a tessellation, which is two coloured, let the two areas of a unit of the tessellation have areas A and B. Assume we work with a union of a large number of tessellating units.

Now consider the folded origami. By hypothesis this is a tessellation, and can be cut into a lattice of fundamental units. In Figure 12, right, the vertices of the lattice are marked by dots. Now cut out one of the fundamental units. If it was unfolded, it will contain a fundamental unit of the original tessellation, though it will generally not be a connected unit. However, it will still have areas coloured, in amounts A and B. Now take the folded parallelogram unit, and cut it into very thin strips. Figure 13 shows a picture of a cross section one of these strips. Because of how the paper is folded, it may contain several components. But, the number of components over any point which this flat origami projects to must be odd. We choose the 2 colouring so that the lowest piece of paper is always the darker colour, with larger area, so that we have a consistent colouring of all strips. Now consider any point in the projection of this strip, x in the figure. Projecting to x, we have an odd number of points; together with the projection point x this is an even number of points, which can be paired up. The folded paper alternates in colour, so we pair up light and dark points, leaving one dark point to pair up with x. In this way, all the light colour points of paper pair up with a dark coloured piece of paper, leaving points in the projection paired up with points on the lowest level of the cross section, which are all of the darker colour. So, we have area A of the darker colour, area B of the lighter; area B of the darker is paired with the lighter colour, leaving $A - B$ of the darker colour to pair with the projection. Hence, if the original area is $A + B$, the folded area is $A - B$. Now we have the ratio of unfolded to folded is $r_2 = (A + B)/(A - B) = (1 + (B/A))/(1 + (B/A)) = (1 + r_1)/(1 - r_1)$ as claimed. ■

6 Summary and Conclusions

This paper has discussed many ways of varying origami tessellations, giving a number of new families. These methods can be applied to many other families of tessellations. A more systematic study would classify all possible continuous variations of waterbomb tessellations and other origami tessellations more generally. The parameter space could be described as a manifold. The techniques described in this paper can be used to continuously deform certain corrugation patterns, in the sense described, to flat tessellations.

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