Continuous Deformations of the Waterbomb Base Tessellation

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Abstract

Eight families of flat foldable continuous deformations of the waterbomb base origami tessellation are presented, as well as a couple of closely related tessellations. Several methods of deforming a tessellation that preserve flatness are explained. The waterbomb tessellation is a corrugation, so we briefly discuss what this means, and how a deformation can transform a corrugation crease pattern to a non-corrugated flat folding origami tessellation crease pattern.

Introduction

Traditionally origami instructions are given by a sequence of folds. However, origami tessellations are typically defined by a *crease pattern* – a collection of vertices, joined by crease lines, either mountain folds (solid line) or valley folds (dashed line). The crease lines indicate how to fold the paper. No other creases are allowed. This description does not completely define a folded origami, because it doesn't tell how far to fold each crease, or how the folded flaps of paper overlap. Nevertheless, for simplicty we ignore this ambiguity.

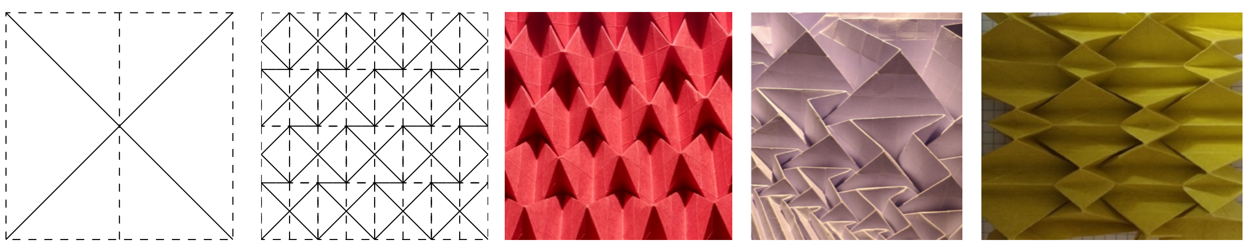


Figure 1: *The waterbomb crease pattern (1); tessellated by Resch (2), folded (3). Variations (4), (5).*

An *origami tessellation* is an origami for which the crease pattern consists of a tessellation of a certain tile, the *unit* of the tessellation. We restrict attention to polygonal units. The edges of the unit are not required to be fold lines, but they often are, which distinguishes tiles in the folded origami. In contrast to a traditional tessellation of tiles which are separate entities, an origami tessellation is folded from a single sheet of paper with no cuts. We mostly consider regular, periodic tilings of a single unit. For these, each unit meets its neighbours in the same way, and the pattern repeats across the plane. Origami tessellations were described by Momotani [11] and Fujimoto [5] in the 1980's, but similar fabric designs go back to the 1800's [13], [16]. The ubiquitous waterbomb tessellation, Figure 1, conceived by Resch, has been utilised in art, design, architecture, and engineering [4], [3], [12]. This is the starting point for the families in this paper.

We restrict attention to *flat foldable* tessellations. This means that when folded the entire origami lies in a single plane. An arbitrary crease pattern won't be flat foldable without additional crease lines. Hull [8] describes the Kawasaki–Justin conditions for flat foldability. In particular, the sum of alternate angles at any vertex is 180°. This can be verified by inspection of the crease patterns in this paper. Full details of flat foldability of the patterns in this paper are omitted for brevity; however, photographs illustrate a number of examples. For further theory see Lang [10] and for more examples see [6] and [7], amongst others.

A distinguished class of flat folding origami tessellations is *corrugations*. In this paper a corrugation is a flat foldable origami with a nonsurjective, infinite to one projection p_2 into the plane in which it lies flat,

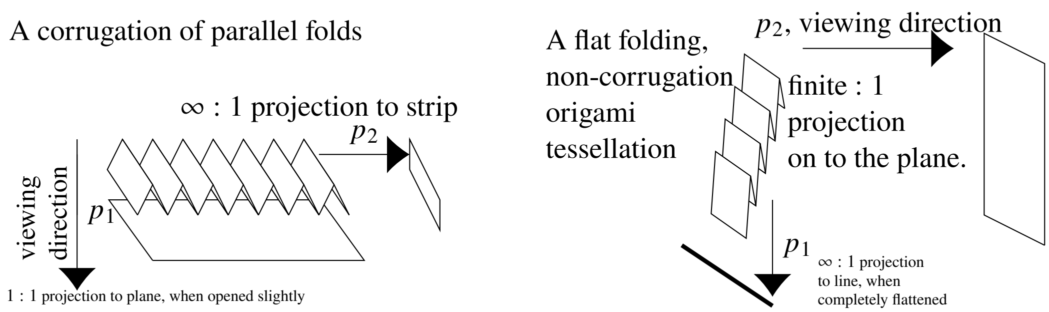


Figure 2: *Projections for viewing origami corrugations and tessellations.*

and when opened slightly a 1 : 1 projection p_1 to a perpendicular plane. Here "infinite" means "arbitrarily large, depending on the paper size". Generally a corrugation is viewed in a perpendicular direction to the flat folding direction. This distinction is shown in Figure 2. The waterbomb base tessellation, our starting point, is a corrugation. However, the typical member of a family of variations in this paper is not a corrugation, though every member is flat foldable (possibly subject to some restrictions on parameters). The fact that some members are corrugations and some are not leads to the question of how to determine this from the crease pattern. An answer can be given by considering a two colouring of the crease pattern. When two coloured, the origami is a corrugation (in our sense) if and only if the two colours have equal area. Details and a proof of the relationship can be found in [15].

My motivating questions concern the relationships between origami tessellations. Given a flat folding origami tessellation, how can it be continuously varied, while remaining a flat foldable tessellation? Can two variations be combined to form a larger family of flat folding tessellations? When is it possible to continuously deform one tessellation to another, in a flat folding family? The difficulty is changing the crease pattern continuously in such a way that the intermediate crease patterns are *all* flat foldable.

Eight Families of Waterbomb Base Origami Tessellations

Parameterised flat origami tessellations based on Archimedian tilings are given in [1], [2], [10] and others. Generally none of these are corrugations. Lamio [9] describes the LamiOri variation on the waterbomb tessellation, consisting entirely of corrugations. In this paper we pass continuously, in the sense that there is no jump in parameters, from non-corrugated tessellations to corrugations, thus including both kinds of tessellation in a single family. We only study variations on the waterbomb base tessellation, but leads one to believe it might be possible to "splurge" other corrugations to form non-corrugated flat origami tessellations. The remainder of the paper is devoted to describing several families of origami tessellations, and how they are constructed. This is not a complete categorisation of variations on the waterbomb base tessellation; work remains to be done to answer the questions mentioned in the introduction. Three of these families are given as interactive JavaScript programs at [14]. Families A-H are continuous deformations of the waterbomb base tessellation shown in Figure 1. Family I is not, as it starts with two different types of unit.

Combining and Extending Two Families

We start with two one-parameter families of origami tessellations and find a two-parameter family containing both of them. In Figure 3 the side length of the original waterbomb unit is fixed to be 1. The x-axis corresponds to the LamiOri variation [9], which splits the vertex at the mountain peak into two vertices, distance x apart, at each end of a mountain ridge. The y-axis corresponds to variations of the "dragon scales" tessellation. The origin corresponds to the waterbomb tessellation, pictured in Figure 3. Can these be combined to form a 2-parameter family of flat foldable origami tessellations? A priori, it is not obvious

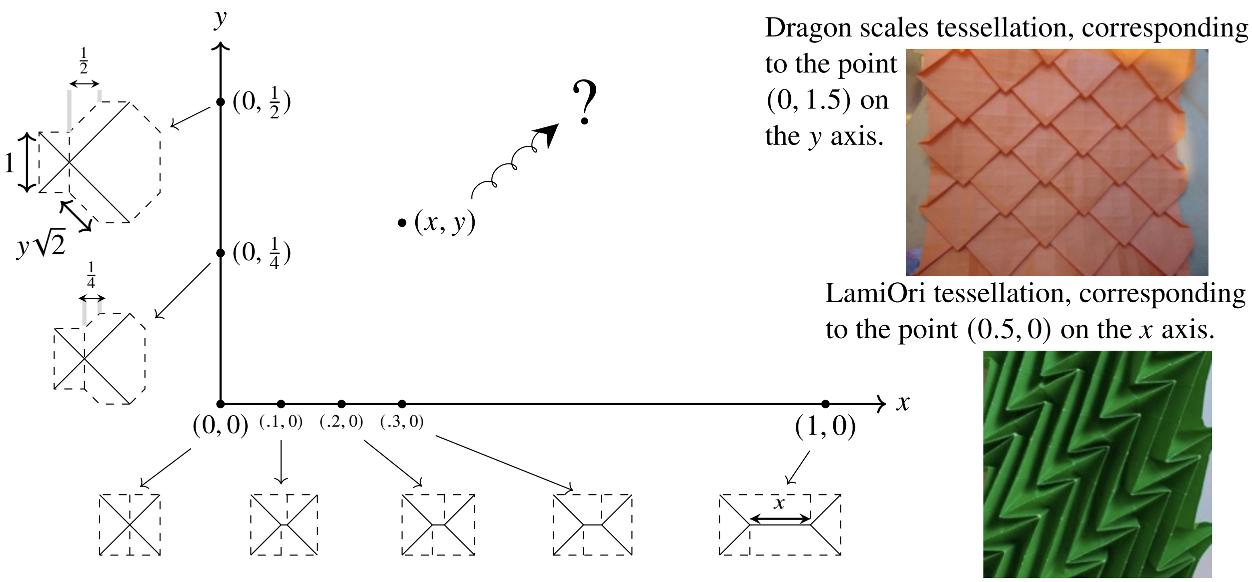


Figure 3: *Points $(x, 0)$ and $(0, y)$ along the axes correspond to the dragon scales and LamiOri variations of the waterbomb tessellation. What tessellation might a point (x, y) correspond to?*

whether this can be done, since generally a perturbation of the crease diagram will not be flat foldable. In this case a solution was found, as in Figure 4, extended as in Figure 5, and labelled Family A in Figure 8a.

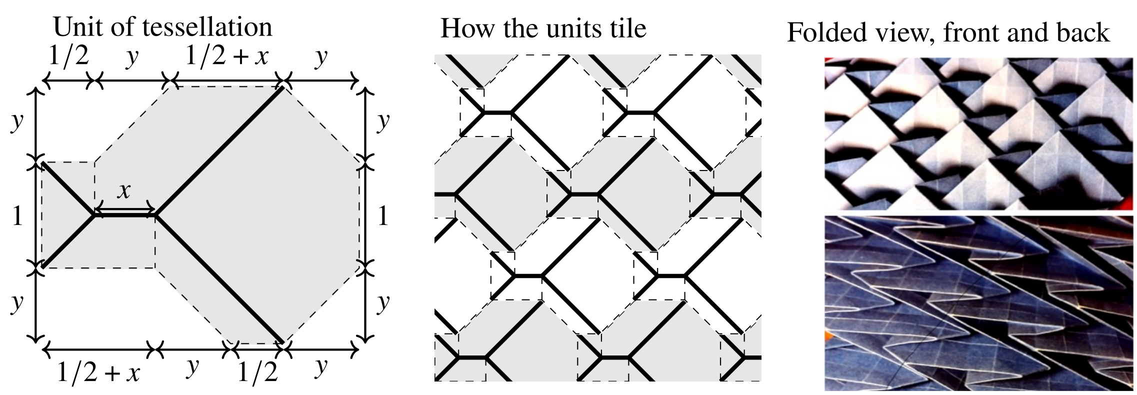


Figure 4: *A two-parameter family incorporating the dragon scales ($x = 0$) and LamiOri ($y = 0$) and waterbomb base ($x = y = 0$) tessellations. In the photograph $(x, y) = (0.5, 0.25)$.*

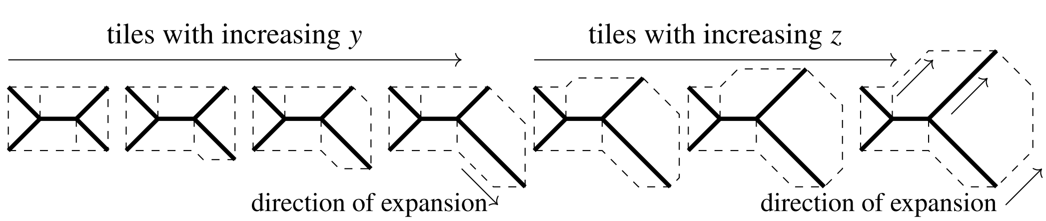


Figure 5: *Adding a parameter to the tile in Figure 4, replacing y by y and z.*

Cut-Move-Rejoin

The appearance of the y and z parameters in Figure 5 can be thought of as a cutting, moving, rejoining procedure, which is now explained in more detail. Consider how the crease patterns in Figure 6 left, contains

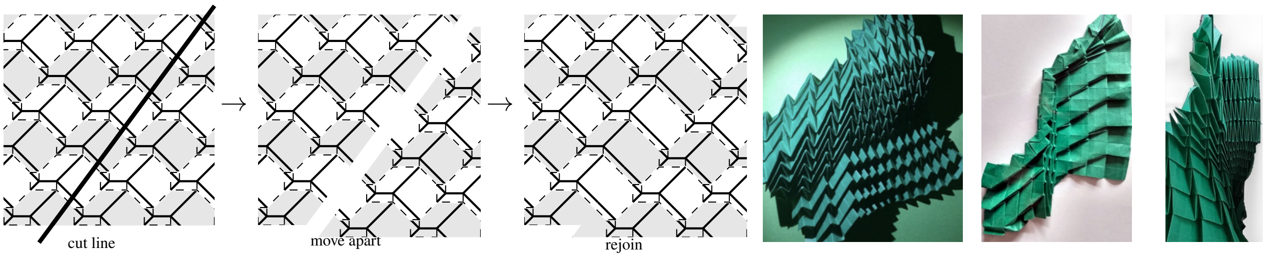


Figure 6: *Modification of a Family A tessellation by cutting through parallel lines. Folded Family C units.*

infinite sets of parallel line segments. If we cut along a line only intersecting a set of mutually parallel line segments, as marked in the figure, and move the two halves of the paper away from each other in the direction of these parallel lines, then reconnect the parallel lines, we obtain a flat folding tessellation consisting of units of the same family, but with different parameter values. This is still flat folding since no angles of the crease pattern change, so the angle condition remains satisfied. The process can be repeated for another cut, possibly in a different direction. Some resulting patterns are shown in Figure 7, with corrugated and non-corrugated

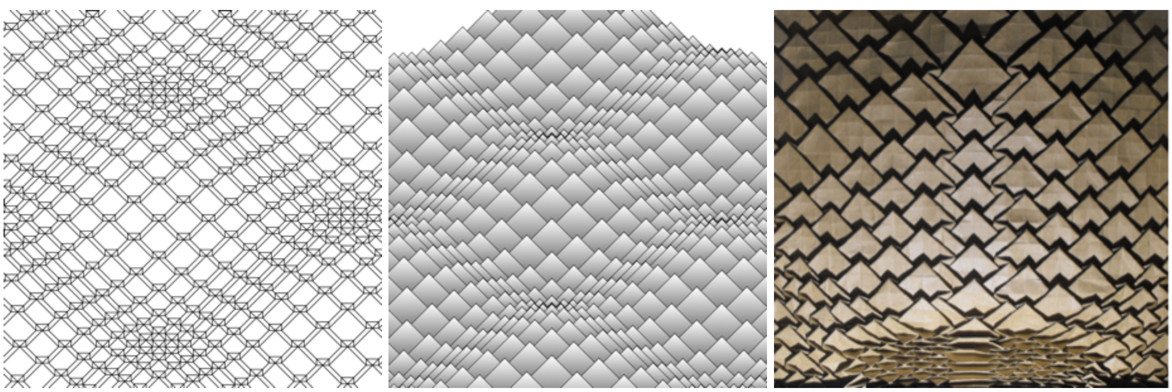
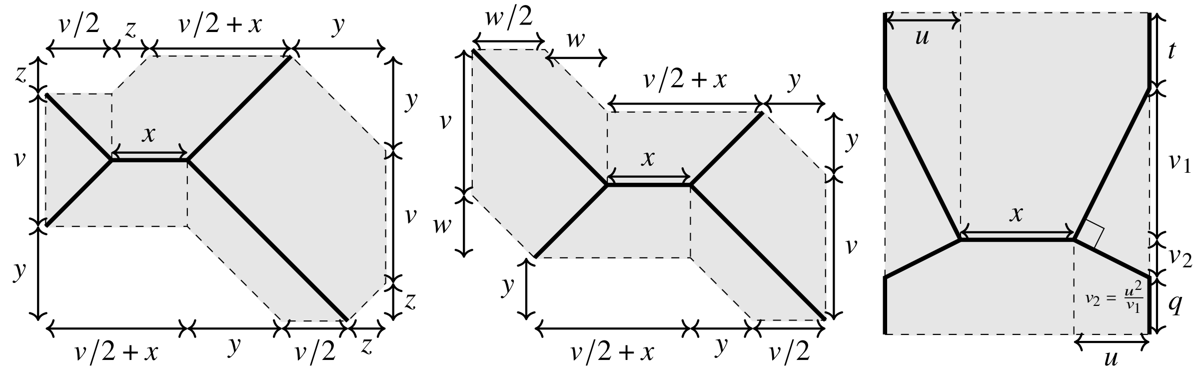


Figure 7: *Tessellations of Family A units obtained via the process in Figure 6. Crease diagram on the left results in middle folded origami; a similar construction results in the figure on the right.*

flat areas in the same origami. This procedure has been applied to obtain parameter w in Family B, Figure 8b, t and q in Family C, Figure 8c, and s, t, q in Families D and E in Figure 11. See also Figure 16. Once the set of parallel lines is visible, e.g., as in the right tile in Figure 5, it is fairly clear to see how the expansion can be made. The difficulty is in finding these parameters from the situation on the left, where the parameter values y and z are zero. Although in Figure 5 it appears this expansion is obtained for an isolated tile, in fact the way this tile meets its neighbours has to be taken into account, so Figure 6 is a more accurate representation of how this parameter is found. The result of the two parameter expansions of Figure 5 is the tile of Family A in Figure 8a. These two parameters are in addition to the LamiOri parameter x, and we add a fourth parameter, v, enabling scaling the whole tile. Family B in Figure 8b is obtained by a similar cut and move method, but expanding in a different direction. Families A and B coincide when $w = z = 0$, and collapse to the waterbomb tessellation when $x = y = z = 0$ and $x = y = w = 0$ respectively. Figure 9 shows tessellations from Families A, B, C with parameters $(v, x, y, z) = (2, 1, 2, 1)$, $(v, w, x, y) = (2, 1, 0, 1)$ and $(u, v_1, q, x, t) = (1, 1, 1, 0, 1)$ respectively. Family C, described below, collapses to the waterbomb case when $x = t = q = u - v_1 = 0$.



(a) *Family A* **(b)** *Family B* **(c)** *Family C*

Figure 8: *Three families of deformations of the waterbomb unit.*

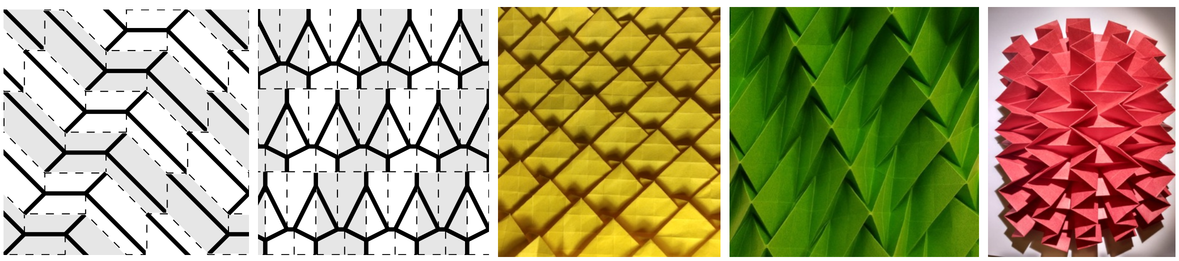


Figure 9: *Tessellations of Families B and C and folded examples of tessellations from Families A, B, C.*

Changing Angles

In Families A and B, tiles were altered by changing lengths, but not angles. A hinged tiling variation of certain tessellations [1], [2] changes angles, but not necessarily lengths. This inspired the Family C variation (Figure 8c) of the LamiOri corrugation, as shown in Figure 10. In Figure 8c the requirement $v_1 v_2 = u^2$

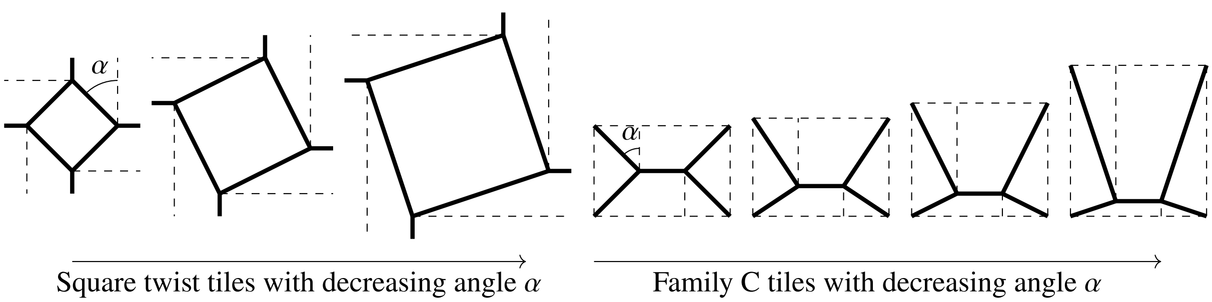


Figure 10: *Comparison between changing an angle in a square twist tessellation and in Family C.*

implies the marked angle is 90°, ensuring flat foldability. Figure 6, right, shows an origami consisting of Family C units with parameters $(u, q) = (1, 0)$, $(v, t) = (1, 1), (1, 0), (2, 0)$ and $x = 0$ or 2. The different areas have different densities (given precisely in [15]), resulting in a three-dimensional effect in the folded piece, when slightly opened, though the origami is flat foldable – up to the paper thickness.

Two other methods of changing the angles are shown in Figure 11. Consider the waterbomb unit as a hexagon, since it has 6 vertices on the perimeter. We deform this to a less square hexagonal tilings of the plane, with crease lines joining the perimeter vertices to a central vertex peak. This peak can be split into two vertices at the ends of a mountain ridge. Families C and D can be combined to form a 6 parameter family by changing the angles labelled 45° in Family D, by the method in Figure 10.

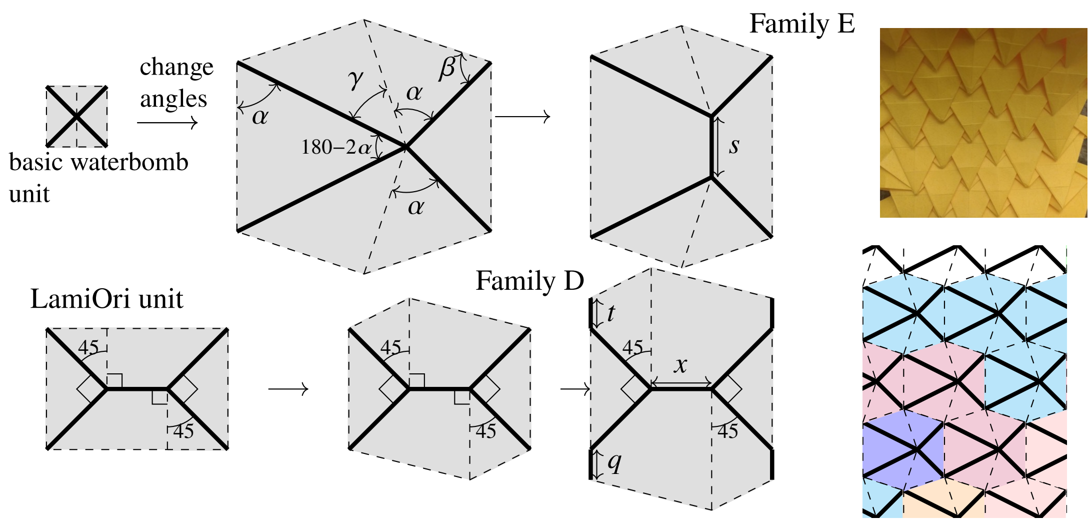


Figure 11: *Families with varying angles. Tessellation and folded origami of a member of Family E shown.*

Stretching Tiles and Splitting Edges

Typically, stretching an origami pattern will not preserve flatness, because stretching does not preserve angles, so the angle condition for flat folding will usually be broken. The stretched unit F_0, Figure 12, is still flat

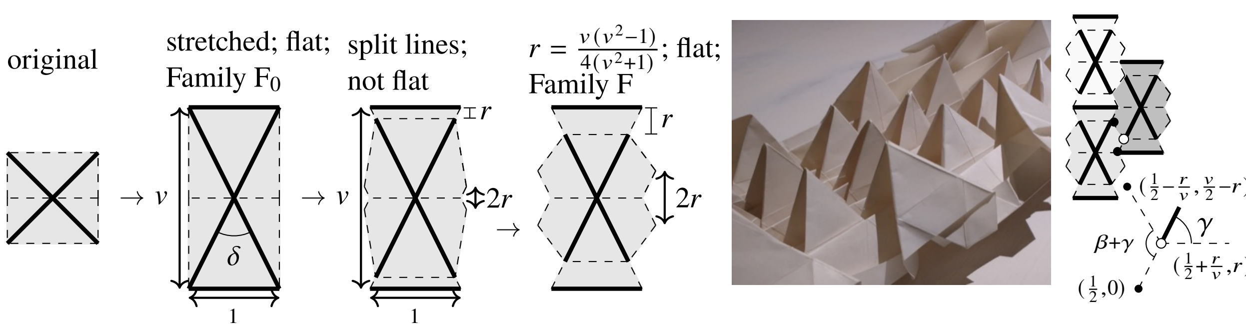


Figure 12: *Stretching a unit, and splitting a crease. An extreme case, with $\delta = 60°$ is shown folded.*

foldable because of the symmetry of the transformation with respect to the angles. However, the flat folding has a curved profile. Instead of projecting to a straight strip, as for a simple case in Figure 2, this corrugation projects to a curved strip. When slightly opened, this forms a cylinder, as in e.g., [17]. To obtain a non-cylindrical tessellation, first split the top and bottom crease lines of the unit. When placed together, a single boundary crease line becomes three alternating parallel crease lines. The result is not flat folding. How these units fit together is shown in Figure 12, right, with a close up of the circled vertex. Coordinates are with respect to the origin at the center of the lower tile. Flat folding requires $\beta + 2\gamma = 180°$, so $\tan(\beta + 2\gamma) = 0$. Since $\tan(\gamma) = v$ and $\tan(\beta) = \frac{v/2-2r}{2r/v}$, the tangent angle sum formula gives $r = \frac{v(v^2-1)}{4(v^2+1)}$. For flat folding $\delta \geq 60°$. When $r = 0$, the union of the three folds becomes a single valley fold, and we return to the basic waterbomb unit.

Multiple Peaks and Different Sized Units

So far, each unit has a single peak or ridge. Arbitrarily many are possible. Figure 13 shows 2 and 3 peaks units. Figure 14 shows a 4 peak unit. All angles are multiples of 45°, so all lengths are determined by those labelled. In Figure 15 units of two sizes are used in one corrugation. Family A deformation of Figure 3 can be applied to each unit, giving the tessellation on the right of Figure 15. The first tessellation is a corrugation, the second is not. The difference is clearer in the pictures where the tessellations, or parts of them, are squashed flat. Related constructions give fractal families of waterbomb units, one of which is shown in Figure 1.

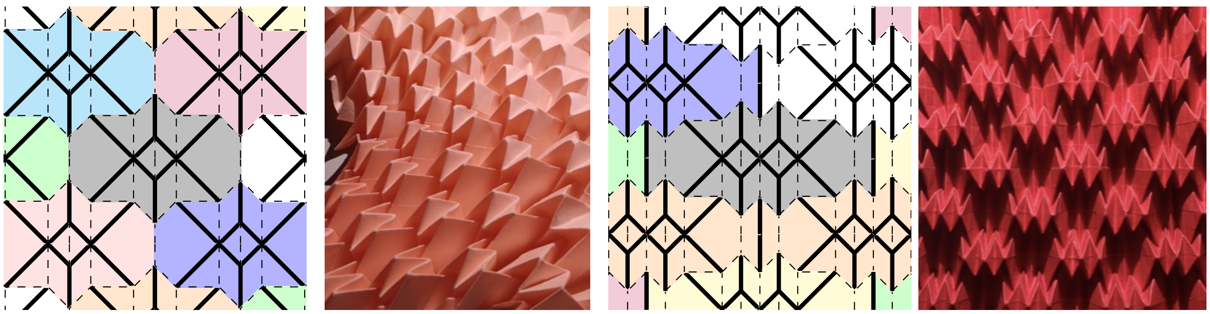


Figure 13: *Families G and H: double and triple peak units.*

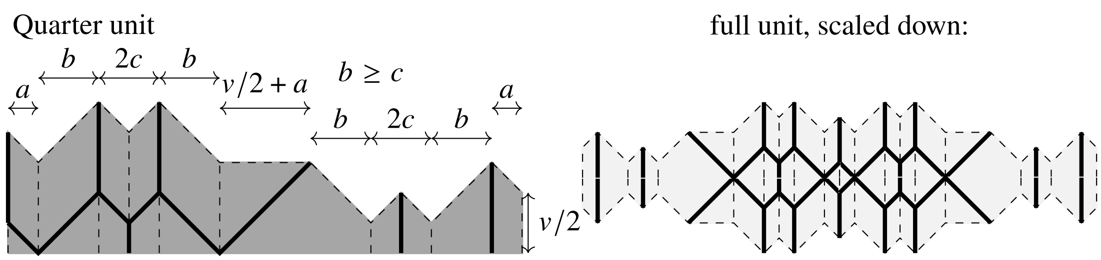


Figure 14: *Units with multiple peaks. Taking $a = b = c = 0$ gives the usual waterbomb tessellation. Families G and H of Figure 13 are obtained with $b = c = 0$ and $a = c = 0$ respectively.*

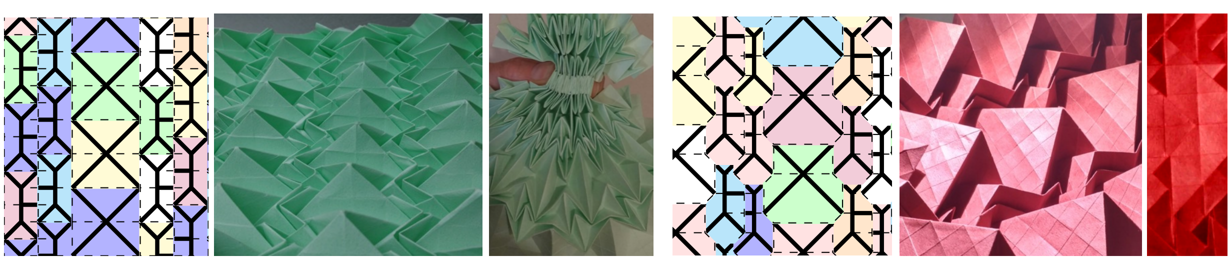


Figure 15: *Family I: Variation of Family A applied to a tessellation of different sized waterbomb units.*

Summary and Conclusions

This paper demonstrated a range of deformations of the waterbomb base tessellation, as a starting point for further exploration. Figure 16 combines several of our parameters into a unit with a missing boundary. Families A–E could be thought of as solutions to the problem "is it possible, for suitable length and angle values, to find a boundary for this tile, to make a tile of a flat origami tessellation?"

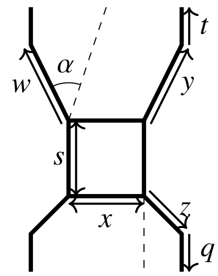


Figure 16: *Can we alter parameters and add a boundary to make this a tile of a flat folding tessellation?*

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