TESSELLATIONS, CORRUGATIONS, AND THE TWO COLOURING RATIO

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An *origami tessellation* is an origami with a tessellation as a crease pattern. An *origami* is defined by a crease pattern, a collection of crease lines, telling us where and how to fold the paper. These lines are mountain or valley folds which I denote by solid or dashed lines respectively.

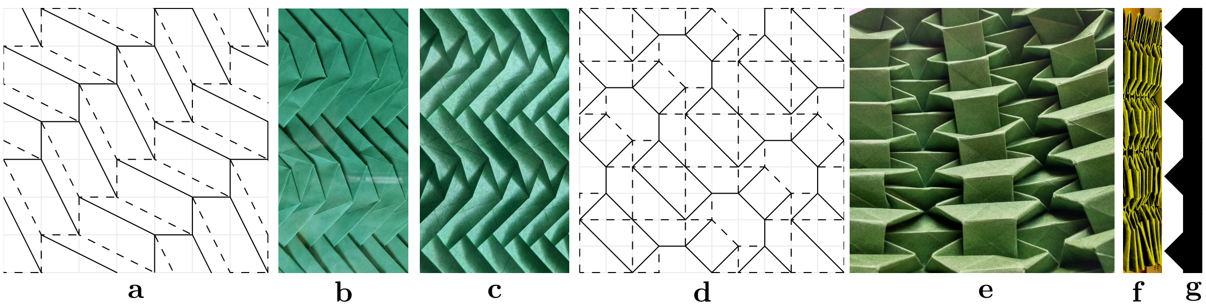


Figure 1. Two flat origami tessellations. A grid aids replication. With backlight, the different layers of paper in the pattern on the left can be seen (**c**). The corrugation on the right has too many layers to see through when folded flat (**g**).

In this paper, all crease patterns are periodic (in two directions) and *flat folding*: after folding, the paper lies in one plane—assuming zero thickness paper, and maximal folding. All lines are straight. Conditions for flatfoldability are given in [2]. Definitions of corrugation vary. We take:

Definition. A *corrugation* is a flat folding origami tessellation, such that the projection in the flat folding direction does not cover the whole plane, and is an ∞ to one projection to its image. This is illustrated in Figure 2. Figure 1 right is a corrugation in this sense.

Can you easily tell from the crease patterns in Figure 1 which is a corrugation? Our main result is a formula for determining from a crease pattern whether or not the resulting flat folding origami is a corrugation. We need two ratios:

Definition. The *colouring ratio* of a flat folding origami is the ratio between the areas of the colours when the pattern is two coloured. Two colourability is proved in [3] .

Definition. The *folding ratio* of a flat folding periodic origami tessellation is the ratio between the initial paper size and the final folded paper size. Suppose the initial square of paper has a given area A, and the final folded origami has an area F. The folding ratio is $R = A/F$. This definition needs to be refined; more details are given in Section 2.

Main Theorem. Given a flat folding repeating origami tessellation with colouring ratio r and folding ratio R, we have:

$$R = f(r) := \left|\frac{1+r}{1-r}\right|. \qquad (*)$$

In particular, the origami is a corrugation if and only if $r = 1$, ie., the colours have equal area.

Note that if r is replaced by $1/r$, the resulting quotient on the left of (*) remains unchanged. The inverse of (*) is $r = f(R)$ or $r = 1/f(R)$, depending on which order the colours are taken in computing the ratio.

1. Notes on the definition and properties of corrugations

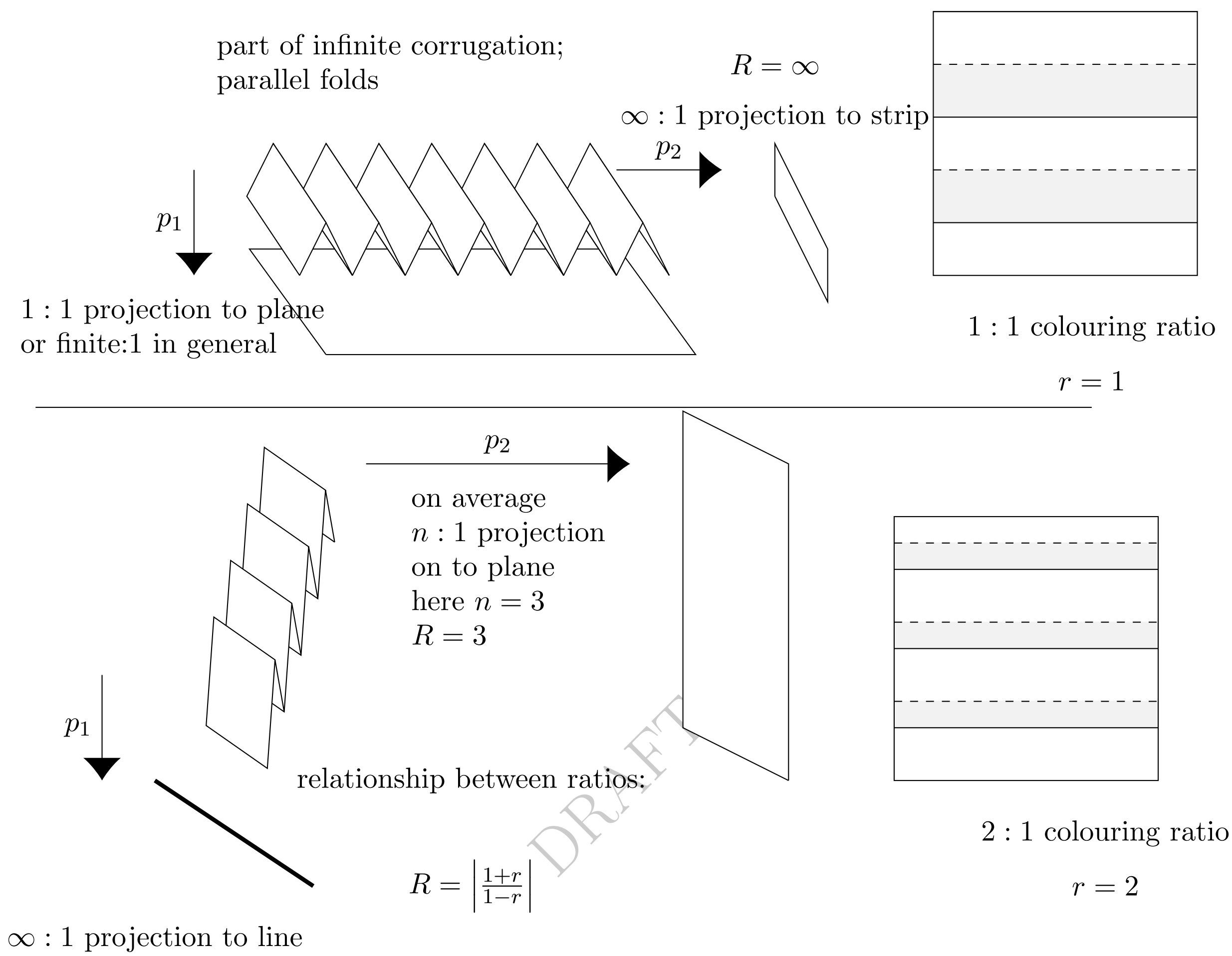


Figure 2. Illustration of the difference between corrugations and noncorrugation flat origami tessellations. A corrugation is generally viewed in direction p_1, whereas a noncorrugation is generally viewed in direction p_2. The coloured crease patterns are also shown, and the two ratios r and R defined above are illustrated.

1.1. **Projections.** Corrugations differ from other tessellations in that they are generally viewed at a perpendicular direction to the flat folding direction, and they are generally viewed slightly opened out, not in the completely flat state. In Figure 1(**e**) the corrugation is viewed in slightly open state, whereas in (**f**) it is squashed flat. Some origamists require the projection p_1 in Figure 2 of a corrugation to the plane to be 1:1. I have not included this in my definition; p_1 is only required to be finite to 1. It would be infinite to 1 if the paper was completely folded flat, so p_1 in Figure 2 refers to a slightly opened out tessellation for the corrugation. It refers to a completely flat folded tessellation for the noncorrugation, even though in the picture, this is also shown slightly open, for clarity, so in the picture, the p_1 for the non corrugation also looks 1:1. The difference in whether the origami is completely folded flat or slightly open is the usual way these origamis are viewed.

The smaller projection image of a corrugation may be important for practical applications. On the other hand, sometimes it is better not to have a corrugation, for example, when origami tessellations are used as variation on pleats in fabric, where we might not want too many layers at

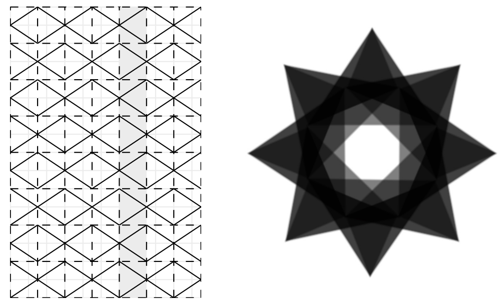


FIGURE 3. Corrugation with finite projection image.

once, and we want to be able to, for example, sit on the fabric without damaging the structure. In [5] I discuss varying parameters of a tessellation to go from a corrugation to a spread out non corrugation, and the theorem of this paper is important to be able to tell how much the paper has been spread out by changing the crease pattern.

1.2. **Light.** Another difference between corrugations and noncorrugations is that for noncorrugations, since the number of layers of paper over any point has a fixed upper bound, even if arbitraily large paper is taken, we may have a nice light pattern, as in Figure 1(**c**), acheived as the number of layers of paper varies. If there are too many layers, or if the number of layers at every point is the same, then, we still won't have a nice light pattern even if it is not a corrugation. For corrugations, as we increase the number of units, the number of layers projecting to a point via p_2 increases, so generally there is no particularly interesting light pattern, other than whether the point in the plane is in the image of p_2 or not, as in Figure 1(**g**).

1.3. **projection image.** A corrugation has the advantage of folding to a much smaller area than a non corrugation. In some examples, the image of the projection is a strip, as in Figure 2, or a unit periodically repeating in a fixed direction, as in Figure 1(**c**). However, it is possible that the image is another shape, which may have finite area. For example, for the stretched waterbombbase tessellation in Figure 3, the image could be a sort of star shape, or an annulus. I have put a program to show different images of this construction in [6]. The 8 pointed star shape shown in the figure is acheived when the side ratio of the basic unit is $\tan(3\pi/16)$, (which is close enough to $2/3$ for that to be a reasonably approximation when practically folded); in the general case we will get an anulus if enough rows are folded. In the image, the paper is shown as different shades of gray, which could be acheived with a backlight if only a thin vertical strip of the pattern (shaded) is folded. The waterbomb base, due to Ron Resch, is a very much studied tessellation, see for example [10] or [1]. I have written about some variations at [5].

2. NOTES ON THE FOLDING RATIO

Suppose we start with a square of paper, side length l, area $A = l^2$ with a origami unit being a rectangle with fixed side lengths a and b (The case where the unit is not a rectangle is similar.) As l tends to infinity, and more units of the tessellation are folded, the ratio A/F tends to a constant. We define this as a limit, because there may possibly be error effects at the edges of the paper, but the larger the paper, the less noticable this effect is. In many cases, the limit ratio can be achieved even for just one unit of the pattern. We could also consider a kind of "folding on a torus". To be more precise, we could say that a crease pattern is just an assignment of solid or dashed lines to white paper. Let's change solid to red and dashed to blue, and so we could consider a crease pattern as a function $c : \mathbf{R}^2 \to \{\text{red,blue,white}\}$. Though this includes crease patterns, it includes all other red, white and blue patterns on the paper, so we would have to improve this to be a definition of a crease pattern. Nevertheless, from a crease pattern we can obtain such a function,

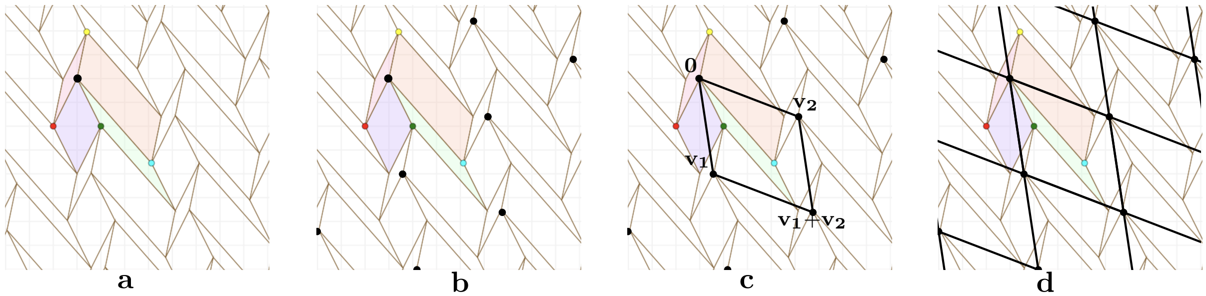


FIGURE 4. Choosing a fundamental unit for the action of a lattice on the tessellation

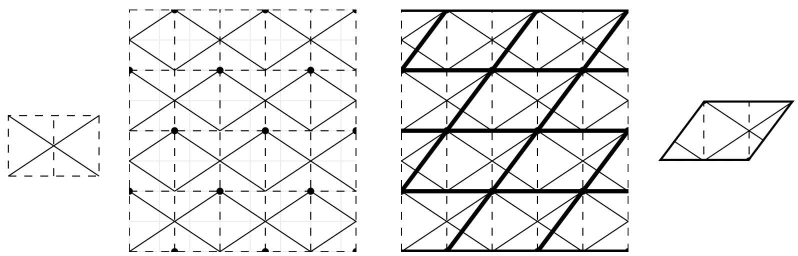


FIGURE 5. A fundamental unit for the stretched waterbomb tessellation (right) compared to original unit of the tessellation (left) (both non unique).

and we can say that the crease pattern is invariant under a discrete subspace $V = \langle \mathbf{v_2}, \mathbf{v_2} \rangle$ of the vector space $\mathbf{R}^2$ if $c(x+v) = c(x)$ for all $x \in \mathbf{R}^2$ and all $v \in V$. Now we can consider the torus $T_1 = \mathbf{R}^2/V$. In the case where the image of V under p_2 of Figure 2 is also a subspace, we have another torus $T_2 = \mathbf{R}^2/V$. the map p_2 factors to $\overline{p}_2 : T_1 \to T_2$, and the folding ratio is given by the ratio between the area of a fundamental unit for T_1 and one for T_2.

Let's review a little about repeating patterns in the plane. It the unit is repeated periodically, then the periods are the basis of the lattice V described in the previous paragraph. The basis is not unique, but a basis can be used to determine a fundamental repeating unit of the pattern. To do this, take a particular point of the pattern, e.g., a corner of a unit, as in Figure 4(**a**). Note that this figure is taken from the page at [7] which is another JavaScript program for investigating origami tessellations, which can also be used to investigate when a flat folding tessellation becomes a corrugation. The black dot is the zero, and the other coloured dots are control points for varying that tessellation. Next mark all the repetitions of this unit, as in Figure 4(**b**). Choose the first copy of this point as $\mathbf{0}$, and two other can be chosen to be the basis of V, as for example in Figure 4(**c**) (non unique choice). Figure 4(**d**) shows how these units then tile the plane. Note that the fundamental unit of the lattice action differs from the unit which might have been used to construct the origami tessellation originally. This is basic linear algebra. Because the tessellation is periodic, the edges of the unit must match up, and they still match up after folding. For example, in Figure 5, the waterbomb base unit of the tessellation is shown on the left. How it is tessellated is in the middle, (see also Figure 3 with a point chosen from each unit, and marked in all the copies, and used to obtain a fundamental unit of the lattice action as shown on the right.

Define our projection p_2 so that $\mathbf{0}$ maps to $\mathbf{0}$ (just compose with translations to acheive this from whatever you start with). Set $\mathbf{w_3} = p_2(\mathbf{v_3})$. Let $\mathbf{w_1} = p_2(\mathbf{v}_1)$, $\mathbf{w_2} = p_2(\mathbf{v}_2)$, and $\mathbf{w_3} = p_2(\mathbf{v}_3)$. These are the corners of the folded fundamental unit.

Label the edges A, B, C, D as in Figure 6. Let their images after the flat folding be $A', B', C'D'$. Edges B is a translated copy of edge A, and edge D is a translated copy of edge C. Since the pattern folds each unit in the same way, we also have B' is a translated copy of A' and D' is a translated

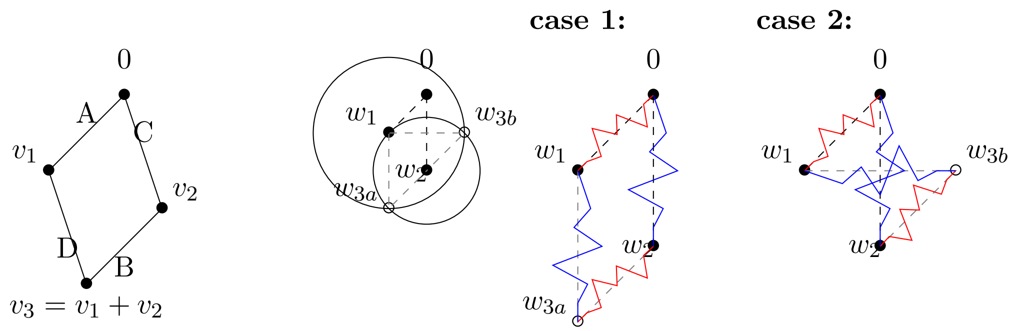


FIGURE 6. Possible image of a fundamental unit of a tessellation. On the left is the original unit. Next, given the images of 0, v_1 and v_2, there are only two possible cases for the image of their sum, w_{3a} and w_{3b}. The possible images on the right are scaled up for clarity, and have an imagined image of the paper edges.

copy of C'. These edges are probably no longer straight line in the image, but the distance from end to end must be the same, so I have put dashed lines in the image picture, just to denote there is some edge from 0 to v_1 and so on, but not the straight line shown. Since the distances of images of opposite sides of the fundamental parallelogram are preserved, we must have that w_3 has one of the two labeled positions in the figure, which are the two possible intersections of the circles C_1 and C_2, with C_1 centred at v_1 and having radius $|v_2|$ and C_2 centred at v_2 and having radius $|v_1|$. Both the illustrated possibilities assume that w_1 and w_2 are linearly independent. This is not the case for the waterbomb base tessellation, for example,

We have the following possible cases:

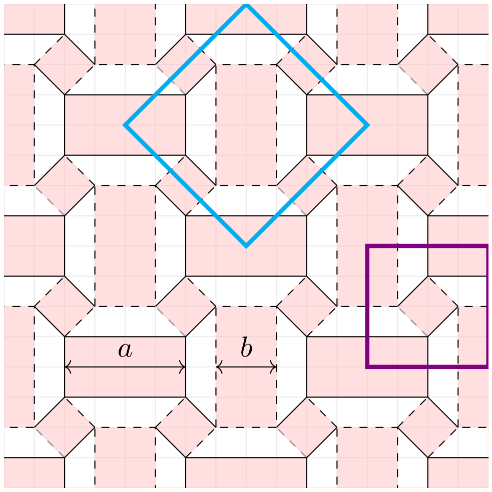
(1) w_1, w_2 are linearly independent and $w_3 = w_1 + w_2$.
(2) w_1, w_2 are linearly independent and $w_3 \neq w_1 + w_2$.
(3) Exactly one of w_1 and w_2 is zero.
(4) $w_1 = w_2 = 0$
(5) $w_2 = \lambda w_1$ for some non zero number λ.

It is intuitively clear that in case 1, the folded unit also tiles the plane, and that we do not have a corrugation. This is the case that this paper is primarily concerned with, when the formula (*) is non trivial. Currently in this **draft** version, I am not addressing the other cases. Case (3) is achieved for the waterbomb tessellation. Case (4) is acheived by for example a crease pattern simply consisting of a square grid, a simple case of the Mira map fold. Will update as soon as possible.

3. EXAMPLES

In order to come up with the formula (*) I investigated the two ratios in examples coming from variations on Momotani's brick wall [4], and later also verified the formula for examples of waterbomb base variations [5]. Some of these examples are as follows.

3.1. **Momotani brick wall examples.** In [8] and [9], I looked at variations on Momotoni's [M] basic brick wall tessellation, shown in Figure 7, where the crease pattern is shown two coloured. This pattern has a rectangular "brick". Suppose this rectangle has side lengths $a \geq b \geq 0$. If we keep the same symmetries of the pattern, then the rest of the pattern is determined by a, b. Properties of the origami that we're intersted in remain unchanged as long as a/b remains unchanged. The ratios r and R described at the beginning of this paper can be easily computed, as in [9], and are shown in the diagram. We can verify that (*) holds for these r and R.



(1) A possible unit for the tessellation is outlined in purple.

(2) A possible choice of fundamental unit is outlined in blue.

(3) The colouring ratio is pink/white $= r = \frac{a^2+b^2}{a^2-b^2}$

(4) The folding ratio is $R = a^2/b^2$

FIGURE 7. crease lines for brick wall[4], with $a = 4, b = 2$

Crease patterns, folded origami, and $a/b, r$ and R

$\frac{3}{2} = 1.5$ $\frac{4}{2} = 2$ $\frac{7}{3} = 2.33$ $\frac{6}{2} = 3$ $\frac{5}{1} = 5$

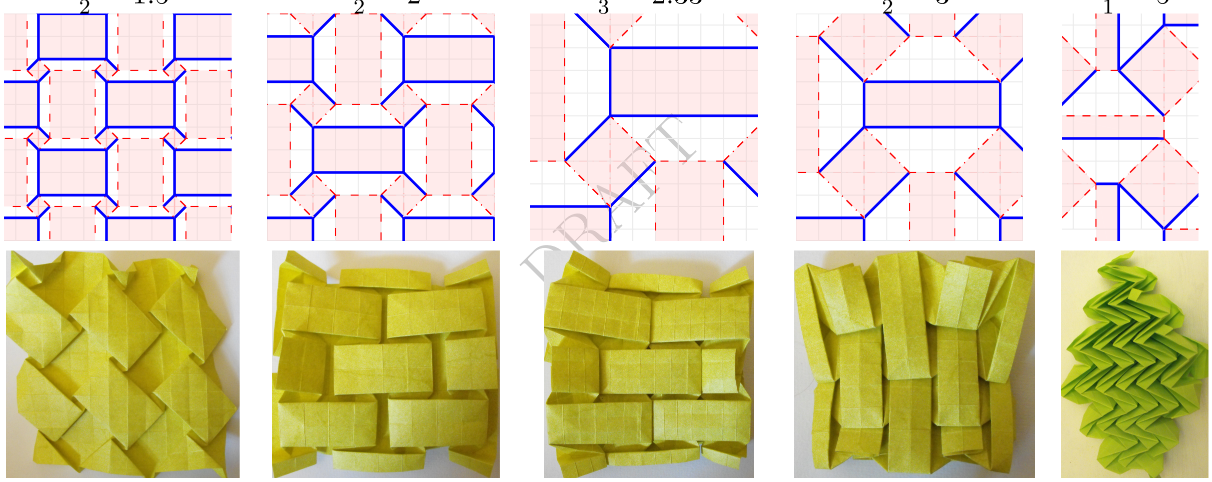


FIGURE 8. Momotani[4] brick wall variations.

Further examples are shown in Figure 8. Note that the case $r = 2$ is the usual brick wall of Momotani [M]. For r with $2 < r \leq 3$, we obtain a flat folding origami, but the bricks overlap. The case $r = 3$ is the limit of being able to fold flat; when $r > 3$, the origami can't fold flat with these crease directions, unless parts on opposite sides of the long side of brick pass through each other. However, if different directions of the creases are chosen, then it does fold flat. The examples with See Figure **??** and Figure **??** for an example, where $r = a/b = 5/1 = 5$.

3.1.1. *Note on isoarea examples.* An *iso area* origami tessellation is a tessellation which has the same appearance from both sides, possibly after some rotation or reflection. One might imagine that iso area origamis are a counter example to our main theorem, because you might imagine that the 2-colouring is reversed when the paper is turned over. However, this does not happen, as is clear from the brick wall pattern, Figure 7, which is an iso area pattern, with bricks appearing on both sides, and also both bricks are the same colour in the two colouring. Even if the colouring was reversed, this is still not a problem, since the formula $f(r)$ is invariant under inverting r.

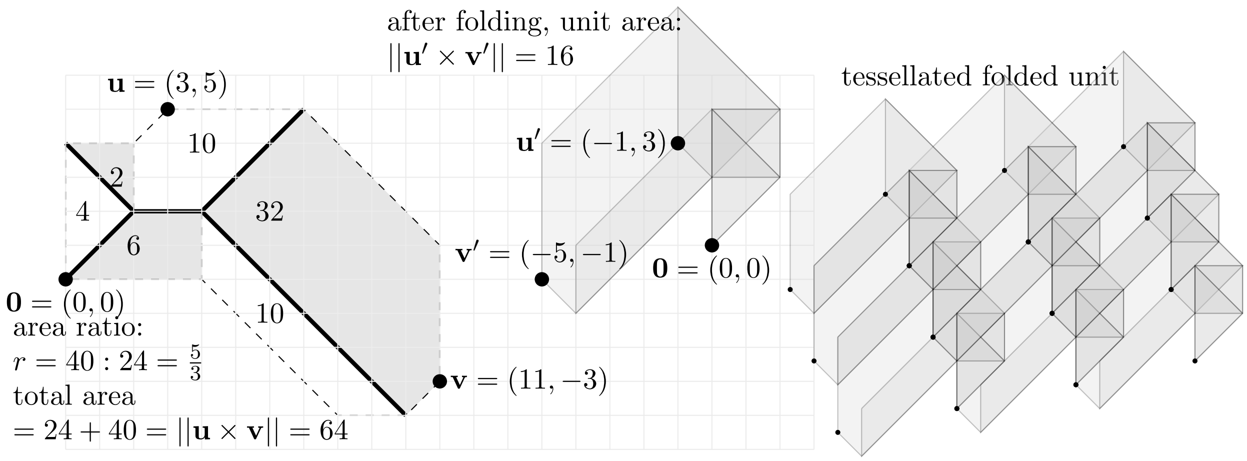


FIGURE 9. Example of ratio relationship for waterbomb variation. This is a unit of Family B, [5], with $a, x, y, z = 4, 2, 4, 1$. The ratio of before to after fold area is $R = 64 : 16 = 4$

3.2. **Waterbomb base examples.** Figure 9 shows a verification of (*) for the case of a variation on the waterbomb base tessellation from [5]. In this picture, I have taken $v = v_1, u = v_2, v' = w_1, u' = w_2$, compared with our previous discussion. Here we obtain $r = 5/3$, $R = 4$, and can check that $(r + 1)/(r - 1) = R$. In this diagram, we have used the fact that the area of a parallelogram defined by vectors u and v is given by the absolute value of their cross product. As mentioned in the discussion of computing the ratio R, we compute the ratio of the fundamental units of the lattices these patterns are invariant under, rather than the area of a single folded unit, since overlaps are not accounted for in the single unit.

4. Proof of Ratio Theorem

Sketch Proof. Given a crease pattern of a tessellation, which is two coloured, let the two areas of a unit of the tessellation have areas A and B. Assume we work with a union of a large number of tessellating units.

Now consider the folded origami. By hypothesis this is a tessellation, and can be cut into a lattice of fundamental units. In Figure 9, right, the vertices of the lattice are marked by dots. Now cut out one of the fundamental units. If it was unfolded, it will contain a fundamental unit of the original tessellation, though it will generally not be a connected unit. However, it will still have areas coloured, in amounts A and B. Now take the folded parallelogram unit, and cut it into very thin strips. Figure 10 shows a picture of a cross section one of these strips. Because of how the paper is folded, it may contain several components. But, the number of components over any point which this flat origami projects to must be odd. We choose the 2 colouring so that the lowest piece of paper is always the darker colour, with larger area, so that we have a consistant colouring of all strips. Now consider any point in the projection of this strip, x in the figure. Projecting to x, we have an odd number of points; togther with the projection point x this is an even number of points, which can be paired up. The folded paper alternates in colour, so we pair up light and dark points, leaving one dark point to pair up with x. In this way, all the light colour points of paper pair up with a dark coloured piece of paper, leaving points in the projection paired up with points on the lowest level of the crossection, which are all of the darker colour. So, we have area A of the darker colour, area B of the lighter; area A of the darker is paired with the lighter colour, leaving $A - B$ to pair with the projection. Hence, if the original area is $A + B$, the folded area is $A - B$. Now we have

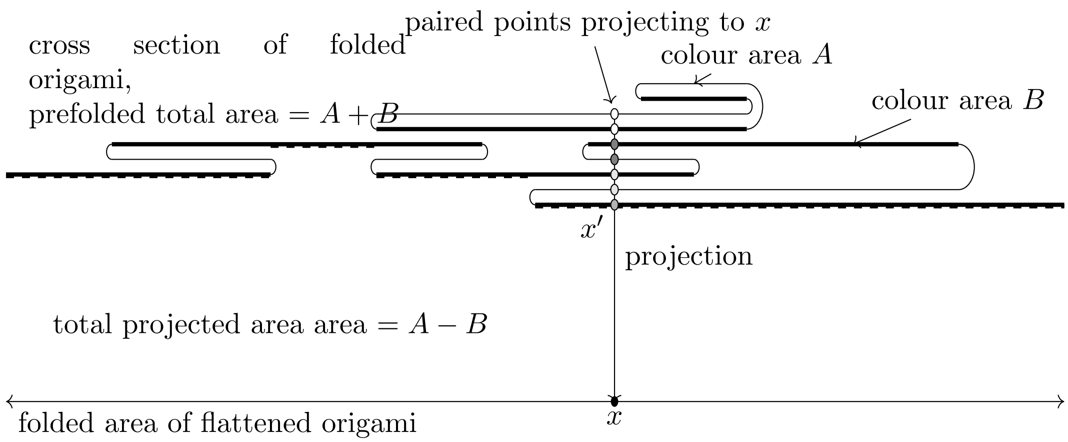


FIGURE 10. Cross section of folded tessellation as explanation of relationship between ratios in Figure 2. The non paired parts of the origami are undelined with dashed lines. These are in bijective correspondence with the projected area below.

the ratio of unfolded to folded is $R = (A+B)/(A-B) = (1+(B/A))/(1+(B/A)) = (1+r)/(1-r)$ as claimed. ■

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